Normal families

Monday, November 6, 2023 12:00 PM

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Theorem (Arzela-Ascoli) (Precompactness criterium)

Let $\mathcal{F}$ be a family of continuous functions $f: \Omega \to \mathbb{C}$ (Ω - a region).
Then any sequence of functions (f_n) from $\mathcal{F}$ contains locally uniformly convergent subsequence (f_{n_k}) if and only if

1) $\mathcal{F}$ is uniformly bounded on compacts: $\forall K \subset \Omega$ - compact $\exists M > 0 : \forall z \in K \ \forall f \in \mathcal{F} \ |f(z)| \le M$.

and

2) $\mathcal{F}$ is uniformly equicontinuous on compacts: $\forall K \subset \Omega$ - compact $\forall \varepsilon > 0 \ \exists \delta > 0 : \forall z_1, z_2 \in K, \ \forall f \in \mathcal{F} : |z_1 - z_2| < \delta \Rightarrow |f(z_1) - f(z_2)| < \varepsilon$.

Proof ($\Downarrow$) If $\mathcal{F}$ is not uniformly bounded on some compact K, then $\exists z_n \in K, f_n \in \mathcal{F} : |f_n(z_n)| \ge n$.

If (f_{n_k}) - uniformly convergent on K subsequence of (f_n), $f_{n_k} \rightrightarrows f$, then $\exists k : \kappa \ge k \Rightarrow |f_{n_\kappa}(z) - f(z)| < 1 \ \forall z \in K \Rightarrow$ $|f_{n_\kappa}(z)|$ is bounded (by $\max\limits_{z \in K} |f(z)| + 1$)

Take $n_\kappa > \max\limits_{z \in K} |f(z)| + 1$ to arrive to contradiction.

If $\mathcal{F}$ is not uniformly equicontinuous then

$\exists \varepsilon > 0 \ \forall n \in \mathbb{N} \ \exists f_n \in \mathcal{F}, z_n, w_n \in K : |z_n - w_n| < \frac{1}{n}, |f_n(z_n) - f_n(w_n)| \ge \varepsilon$.

Let $f_{n_k} \rightrightarrows f$, f is uniformly continuous, so $\exists \delta > 0 : |z - w| < \delta \Rightarrow |f(z) - f(w)| < \frac{\varepsilon}{3}$

Also $\exists k : \kappa > k \Rightarrow |f_{n_\kappa}(z) - f(z)| < \frac{\varepsilon}{3}$. So if we pick $\frac{1}{n_\kappa} < \delta$, we get

$|f_{n_k}(z_{n_k}) - f_{n_k}(w_{n_k})| \le |f_{n_k}(z_{n_k}) - f(z_{n_k})| + |f(z_{n_k}) - f(w_{n_k})| + |f(w_{n_k}) - f_{n_k}(w_{n_k})| < \varepsilon$.

Contradiction!

($\Uparrow$) Let $(\zeta_k) \subset \Omega$ be a dense sequence of points (i.e. $\mathrm{Clos}\{\zeta_k\} = \mathrm{Clos}(\Omega)$).

Let $(f_n) \subset \mathcal{F}$ be a sequence.

Since $(f_n(\zeta_1))$ is a bounded sequence, it has a convergent subsequence $(f_{n,1}(\zeta_1))$

Since $(f_{n,1}(\zeta_2))$ is a bounded sequence, it has a convergent subsequence $(f_{n,2}(\zeta_2))$. Also, $(f_{n,2}(\zeta_1))$ converge (as a subsequence of $(f_{n,1}(\zeta_1))$).

...

Repeat to get $(f_{n,m})_{n=1}^{\infty}$, such that $\forall j \le m$, $(f_{n,m}(\zeta_j))_{n=1}^{\infty}$ is a convergent sequence.

Define g := f Then $\forall j$ $(g(\zeta_j))$ is convergent

$(f_{n,m}(\zeta_i))_{n=1}^{\infty}$ is a convergent sequence.

Define $g_k := f_{k,k}$. Then $\forall i$ $(g_k(\zeta_i))$ is convergent,
since $\forall i$ for $k \geq i$, $(g_k(\zeta_i)) = (f_{k,k}(\zeta_i))$ is a subsequence of convergent $(f_{n,i}(\zeta_i))_{n=1}^{\infty}$.

Let us prove that g_k converges locally uniformly in Ω.

By the definition of local uniform convergence

Local Uniform Convergence

we only need to prove that $\forall z \in \Omega\ \forall \varepsilon > 0\ \exists \delta(\varepsilon, z) > 0, N(\varepsilon, z)$:
$n, m > N(\varepsilon, z) \Rightarrow \forall w \in B(z, \delta)\ |g_n(w) - g_m(w)| < \varepsilon$.

Bonus (+1 pt). Mistake in Ahlfors in this proof

Fix $\varepsilon > 0$, $z \in \Omega$. Let $r < dist(z, \partial\Omega)$

$\mathcal{F}$ is equicontinuous on compact $\overline{B(z,r)} \subset \Omega$, so

$\exists r > \delta > 0$: $|w_1 - w_2| < 2\delta$, $w_1, w_2 \in B(z, r) \Rightarrow \forall f \in \mathcal{F}$, $|f(w_1) - f(w_2)| < \frac{\varepsilon}{3}$.

Consider $B(z, \delta)$. $\exists k: \zeta_k \in B(z, \delta)$ – dense!

$\exists N$: $n, m > N$ $|g_n(\zeta_k) - g_m(\zeta_k)| < \frac{\varepsilon}{3}$.

Then $\forall w \in B(z, \delta)$: $|g_n(w) - g_m(w)| \leq \underbrace{|g_n(w) - g_n(\zeta)|}_{< \frac{\varepsilon}{3} \text{ (since } |w - \zeta| < 2\delta)} + \underbrace{|g_n(\zeta) - g_m(\zeta)|}_{< \frac{\varepsilon}{3} \ (n, m > N)} + \underbrace{|g_m(\zeta) - g_m(w)|}_{< \frac{\varepsilon}{3} \ (|w - \zeta| < 2\delta)} < \varepsilon$.

Paul Montel

Def Let Ω be a region, $\mathcal{F} \subset A(\Omega)$ – a family of analytic functions is called normal if $\forall$ sequence $(f_n) \subset \mathcal{F}$ $\exists$ a subsequence (f_{n_k}) converging locally uniformly.

Theorem (Montel) $\mathcal{F}$ is normal iff

Theorem (Montel) $\mathcal{F}$ is normal iff it is uniformly bounded on compacts.

Proof. If $\mathcal{F}$ is not uniformly bounded on some $K \subset \Omega$ - compact then $\exists (f_n) \subset \mathcal{F}$, $z_n \in \Omega$: $|f_n(z_n)| \to \infty$. In particular, for any subsequence $|f_{n_k}(z_{n_k})| \to \infty$, so for any $f \in \mathcal{A}(\Omega)$, $\sup_{z \in K} |f_{n_k}(z) - f(z)| \geq |f_{n_k}(z_{n_k}) - f(z_{n_k})| \to \infty$ - does not converge!

Let $\mathcal{F}$ be uniformly bounded on compacts. By Arzela Theorem, we need to prove equicontinuity on compacts.

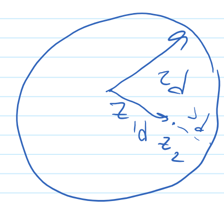
let $K \subset \Omega$ - compact. $z \to \text{dist}(z, \partial\Omega)$ - continuous on K, so it reaches minimum.

So $\exists\, d > 0$: $\forall z \in K$ $\text{dist}(z, \partial\Omega) > 4d \Rightarrow \overline{B(z, 3d)} \subset \Omega$.

Let $F := \{z \in \mathbb{C} : \text{dist}(z, K) \leq 2d\} \subset \Omega$, closed, bounded, so F is compact.

Let $M := \max\{|f(z)| : z \in F\}$.

If $z_1, z_2 \in K$, $|z_1 - z_2| < d$, consider $C_{2d} = \{z : |z - z_1| = 2d\}$, positively oriented. Then $n(C_{2d}, z_1) = n(C_{2d}, z_2) = 1$. $C_{2d} \subset F$.



$$f(z_1) = \frac{1}{2\pi i} \oint_{C_{2d}} \frac{f(\zeta)}{\zeta - z_1}\, d\zeta \qquad f(z_2) = \frac{1}{2\pi i} \oint_{C_{2d}} \frac{f(\zeta)}{\zeta - z_2}\, d\zeta$$

$$f(z_1) - f(z_2) = \frac{z_1 - z_2}{2\pi i} \oint_{C_{2d}} \frac{f(\zeta)}{(\zeta - z_1)(\zeta - z_2)}\, d\zeta$$

So $|f(z_1) - f(z_2)| \leq \frac{1}{2\pi} |z_1 - z_2| \,\text{length}(C_{2d}) \cdot \frac{M}{2d \cdot d} = |z_1 - z_2| \frac{M}{d}$.

(since $|\zeta - z_1| = 2d$, $|\zeta - z_1| \geq |\zeta - z_2| - |z_1 - z_2| \geq d$)

So for $\varepsilon > 0$, if $\delta = \min\left(d, \frac{\varepsilon d}{M}\right)$, then

$|z_1 - z_2| < \delta \Rightarrow |z_1 - z_2| < d$ so $|f(z_1) - f(z_2)| < \delta \frac{M}{d} = \varepsilon$ ∎

Corollary (Montel's convergence criterium).

Assume $(f_n) \subset \mathcal{A}(\Omega)$ is locally uniformly bounded. If every convergent subsequence (f_{n_k}) of (f_n) converges locally uniformly to f, then $f_n \to f$ locally uniformly.

Let f_n does not converge to f locally. It means $\exists K \subset \Omega$ - compact, $\varepsilon > 0$:

$\forall N\ \exists n>N: \sup_{z\in K}|f_n(z)-f(z)| \geq \varepsilon$. Take n_1: $\sup_{z\in K}|f_{n_1}(z)-f(z)| \geq \varepsilon$.

Take $n_2>n_1$: $\sup_{z\in K}|f_{n_2}(z)-f(z)|>\varepsilon$

Construct recursively $n_k>n_{k-1}$: $\sup_{z\in K}|f_{n_k}(z)-f(z)|>\varepsilon$.

Then $g_k := f_{n_k}$ is locally uniformly bounded.

So it has a subsequence (g_{k_ℓ}) which converges on K to g.

But $\sup_{z\in K}|g(z)-f(z)| = \lim_{\ell\to\infty}\sup|g_{k_\ell}(z)-f(z)| \geq \varepsilon$, so $g\neq f$.

But g_{k_ℓ} – subsequence of (g_k), which is a subsequence of (f_n).

So (g_{k_ℓ}) – convergent subsequence of (f_n) which does not converge to f – contradiction! ∎